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Linear Algebra and its Applications 429 (2008) 2227–2238

LINEAR ALGEBRA AND ITS APPLICATIONS

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Which 2-hyponormal 2-variable weighted shifts are subnormal?☆

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Received 6 July 2007; accepted 13 June 2008
Available online 3 August 2008
Submitted by C.-K. Li

Abstract

It is well known that a 2-hyponormal unilateral weighted shift with two equal weights must be flat, and therefore subnormal. By contrast, a 2-hyponormal 2-variable weighted shift which is both horizontally flat and vertically flat need not be subnormal. In this paper we identify a large class $\mathscr{S}$ of flat 2-variable weighted shifts for which 2-hyponormality is equivalent to subnormality. One measure of the size of $\mathscr{S}$ is given by the fact that within $\mathscr{S}$ there are hyponormal shifts which are not subnormal.

AMS classification: Primary 47B20, 47B37, 47A13; Secondary 44A60, 47-04, 47A20, 28A50

Keywords: Jointly hyponormal pairs; 2-Hyponormal pairs; Subnormal pairs; 2-Variable weighted shifts; Flatness

1. Statement of the main results

The lifting problem for commuting subnormals (LPCS) asks for necessary and sufficient conditions for a pair of subnormal operators on Hilbert space to admit commuting normal extensions.

☆ Research partially supported by NSF Grants DMS-0099357 and DMS-0400741; the second author was also partially supported by a Korea Research Foundation Grant KRF-2007-331-C00013, funded by the Korean Government (MOEHRD, Basic Research Promotion Fund).

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doi:10.1016/j.laa.2008.06.017

It is well known that the commutativity of the pair is necessary but not sufficient [1,18–20], and it has recently been shown that the joint hyponormality of the pair is necessary but not sufficient [13]. Our previous work [9,10,13–15,24,25] has revealed that the nontrivial aspects of LPCS are best detected within the class $\mathfrak{H}_0$ of commuting pairs of subnormal operators; we thus focus our attention on this class. The class of subnormal pairs on Hilbert space will be denoted by $\mathfrak{H}_\infty$, and for an integer $k \geqslant 1$ the class of k-hyponormal pairs in $\mathfrak{H}_0$ will be denoted by $\mathfrak{H}_k$. Clearly, $\mathfrak{H}_\infty \subseteq \cdots \subseteq \mathfrak{H}_k \subseteq \cdots \subseteq \mathfrak{H}_2 \subseteq \mathfrak{H}_1 \subseteq \mathfrak{H}_0$; the main results in [13,9] show that these inclusions are all proper. It is then natural to look for subclasses of $\mathfrak{H}_0$ on which subnormality and k-hyponormality agree, that is, classes on which subnormality can be detected with a matricial test.

In this paper, we identify a large class $\mathscr{S} \subseteq \mathfrak{H}_0$ on which 2-hyponormality and subnormality agree, that is, $\mathscr{S} \bigcap \mathfrak{H}_2 = \mathscr{S} \bigcap \mathfrak{H}_\infty$. Concretely, $\mathscr{S}$ consists of all 2-variable weighted shifts $\mathbf{T} \equiv (T_1, T_2) \in \mathfrak{H}_0$ such that $\alpha_{(k_1,0)} = \alpha_{(k_1+1,0)}$ and $\beta_{(0,k_2)} = \beta_{(0,k_2+1)}$ for some $k_1 \geqslant 1$ and $k_2 \geqslant 1$, where α and β denote the weight sequences of T_1 and T_2, respectively. One measure of the size of $\mathscr{S}$ is given by the fact that hyponormality and subnormality do not agree on $\mathscr{S}$, that is $\mathscr{S} \bigcap \mathfrak{H}_\infty \subsetneq \mathscr{S} \bigcap \mathfrak{H}_1$. Thus, $\mathscr{S}$ consists of nontrivial shifts for which 2-hyponormality and subnormality are equivalent, but for which hyponormality and 2-hyponormality are different; that is, $\mathscr{S}$ is small enough to ensure that 2-hyponormality implies subnormality, but large enough to separate hyponormality from subnormality.

Each *hyponormal* shift in $\mathscr{S}$ is *flat*, that is, $\alpha_{(k_1,k_2)} = \alpha_{(1,1)}$ and $\beta_{(k_1,k_2)} = \beta_{(1,1)}$ for all k_1, $k_2 \geqslant 1$. As a result, each hyponormal shift in $\mathscr{S}$ belongs to the class $\mathscr{TC}$ of shifts whose core is of tensor form (cf. Definition 2.8); $\mathscr{TC}$ is a class that we have studied in detail in [11]. Previously, in [10] we had proved that there exist 2-variable weighted shifts in $\mathscr{TC}$ which are hyponormal but not subnormal. By contrast, the 2-hyponormal shifts in $\mathscr{S}$ are automatically subnormal.

We prove our main results by combining the 15-point Test for 2-hyponormality [9] with the Subnormal Backward Extension Criterion [13] and Smul'jan's Test for positivity of operator matrices [21]. As a first step, we prove a propagation result for 2-hyponormal 2-variable weighted shifts (Theorem 2.11). We then seek conditions to guarantee the subnormality of $\mathbf{T} \equiv (T_1, T_2) \in \mathfrak{H}_2$.

Our main results follow; first, we need a definition.

Definition 1.1. $\mathscr{S} := \{\mathbf{T} \equiv (T_1, T_2) \in \mathfrak{H}_0 \colon \alpha_{(k_1,0)} = \alpha_{(k_1+1,0)}$ and $\beta_{(0,k_2)} = \beta_{(0,k_2+1)}$ for some $k_1 \geqslant 1$ and $k_2 \geqslant 1\}$. (Here α and β denote the weight sequences of T_1 and T_2, respectively; cf. Section 2.)

Theorem 1.2. $\mathscr{S} \bigcap \mathfrak{H}_2 = \mathscr{S} \bigcap \mathfrak{H}_\infty$.

The generic form of the weight diagram of a hyponormal 2-variable weighted shift in $\mathscr{S}$ is given in Fig. 2(ii). Using that notation, we can sharpen Theorem 1.2 as follows.

Theorem 1.3. *Let* $\mathbf{T} \equiv (T_1, T_2) \in \mathscr{S}$, *with weight diagram given by Fig.* 2(ii), *and assume that* $\mathbf{T}$ *is* 2-*hyponormal*. *Then* $\mathbf{T}$ *is subnormal*, *with Berger measure given as*

$$\mu = \frac{1}{b^2}\Big\{[b^2(1-x^2) - y^2(1-a^2)]\delta_{(0,0)} + y^2(1-a^2)\delta_{(0,b^2)} + (b^2x^2 - a^2y^2)\delta_{(1,0)} + a^2y^2\delta_{(1,b^2)}\Big\}.$$

Theorem 1.4. $\mathscr{S} \bigcap \mathfrak{H}_\infty \subsetneq \mathscr{S} \bigcap \mathfrak{H}_1$.

Remark 1.5. Hyponormality alone does not imply flatness. While it is true that in the presence of hyponormality the Six-point Test creates L-shaped propagation (i.e., $\alpha_{\mathbf{k}+\boldsymbol{\varepsilon}_1} = \alpha_{\mathbf{k}} \Longrightarrow \alpha_{\mathbf{k}} = \alpha_{\mathbf{k}+\boldsymbol{\varepsilon}_2}$ and $\beta_{\mathbf{k}} = \beta_{\mathbf{k}+\varepsilon_1}$; cf. [15, Proof of Theorem 3.3]), without horizontal propagation (as guaranteed by the quadratic hyponormality of T_1) this L-propagation does not result in vertical propagation, needed to eventually lead to flatness. The same phenomenon arises in one variable, where hyponormality is a very soft condition ($\alpha_k \leqslant \alpha_{k+1}$ for all $k \geqslant 0$), while 2-hyponormality is quite rigid. The work in [6] (extending the ideas in [22]) revealed that, for unilateral weighted shifts with two equal weights, 2-hyponormality and subnormality are identical notions. In two variables, however, the analogous result does not hold, as the present work shows.

2. Notation and preliminaries

For $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$ a bounded sequence of positive real numbers (called *weights*), let $W_\alpha \equiv \mathrm{shift}(\alpha_0, \alpha_1, \ldots) : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be the associated *unilateral weighted shift*, defined by $W_\alpha e_n := \alpha_n e_{n+1}$ (all $n \geqslant 0$), where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$. Two special cases of significant interest are $U_+ := \mathrm{shift}(1, 1, \ldots)$ (the (unweighted) unilateral shift) and $S_a := \mathrm{shift}(a, 1, 1, \ldots)$ $(0 < a < 1)$. The *moments* of α are given as

$$\gamma_k \equiv \gamma_k(\alpha) := \begin{Bmatrix} 1 & \text{if } k = 0 \\ \alpha_0^2 \cdots \alpha_{k-1}^2 & \text{if } k > 0 \end{Bmatrix}.$$

It is easy to see that W_α is never normal, and that it is hyponormal if and only if $\alpha_0 \leqslant \alpha_1 \leqslant \cdots$. Similarly, consider double-indexed positive bounded sequences $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} \in \ell^\infty(\mathbb{Z}_+^2)$, $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2 := \mathbb{Z}_+ \times \mathbb{Z}_+$ and let $\ell^2(\mathbb{Z}_+^2)$ be the Hilbert space of square-summable complex sequences indexed by $\mathbb{Z}_+^2$. We define the 2-*variable weighted shift* $\mathbf{T} \equiv (T_1, T_2)$ by

$$T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1}$$
$$T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2},$$

where $\boldsymbol{\varepsilon}_1 := (1, 0)$ and $\boldsymbol{\varepsilon}_2 := (0, 1)$. Clearly

$$T_1 T_2 = T_2 T_1 \Longleftrightarrow \beta_{\mathbf{k}+\varepsilon_1} \alpha_{\mathbf{k}} = \alpha_{\mathbf{k}+\varepsilon_2} \beta_{\mathbf{k}} \quad (\text{all } \mathbf{k} \in \mathbb{Z}_+^2). \tag{2.1}$$

Trivially, a pair of unilateral weighted shifts W_α and W_β gives rise to a 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$, if we let $\alpha_{(k_1,k_2)} := \alpha_{k_1}$ and $\beta_{(k_1,k_2)} := \beta_{k_2}$ (all $k_1, k_2 \in \mathbb{Z}_+$). In this case, $\mathbf{T}$ is subnormal (resp. hyponormal) if and only if so are T_1 and T_2; in fact, under the canonical identification of $\ell^2(\mathbb{Z}_+^2)$ with $\ell^2(\mathbb{Z}_+) \bigotimes \ell^2(\mathbb{Z}_+)$, we have $T_1 \cong I \bigotimes W_\alpha$ and $T_2 \cong W_\beta \bigotimes I$, so $\mathbf{T}$ is also doubly commuting.

We now recall a well known characterization of subnormality for single variable weighted shifts, due to C. Berger (cf. [4, III.8.16]), and independently established by Gellar and Wallen [16]): W_α is subnormal if and only if there exists a probability measure ξ supported in $[0, \|W_\alpha\|^2]$, with $\|W_\alpha\|^2 \in \operatorname{supp} \xi$, and such that $\gamma_k(\alpha) := \alpha_0^2 \cdots \alpha_{k-1}^2 = \int s^k \, \mathrm{d}\xi(s)$ $(k \geqslant 1)$.

We also recall the notion of moment of order $\mathbf{k}$ for a pair (α, β) satisfying (2.1). Given $\mathbf{k} \in \mathbb{Z}_+^2$, the moment of (α, β) of order $\mathbf{k}$ is

$$\gamma_{\mathbf{k}} \equiv \gamma_{\mathbf{k}}(\alpha, \beta) := \left\{ \begin{array}{ll} 1 & \text{if } \mathbf{k} = 0 \\ \alpha^2_{(0,0)} \cdots \alpha^2_{(k_1-1,0)} & \text{if } k_1 \geqslant 1 \text{ and } k_2 = 0 \\ \beta^2_{(0,0)} \cdots \beta^2_{(0,k_2-1)} & \text{if } k_1 = 0 \text{ and } k_2 \geqslant 1 \\ \alpha^2_{(0,0)} \cdots \alpha^2_{(k_1-1,0)} \cdot \beta^2_{(k_1,0)} \cdots \beta^2_{(k_1,k_2-1)} & \text{if } k_1 \geqslant 1 \text{ and } k_2 \geqslant 1 \end{array} \right\}. \tag{2.2}$$

We remark that, due to the commutativity condition (2.1), $\gamma_{\mathbf{k}}$ can be computed using any nondecreasing path from (0,0) to (k_1, k_2). Moreover, $\mathbf{T}$ is subnormal if and only if there is a regular Borel probability measure μ defined on the 2-dimensional rectangle $R = [0, a_1] \times [0, a_2]$ ($a_i := \|T_i\|^2$) such that

$$\gamma_{\mathbf{k}} = \int \int_R s^{k_1} t^{k_2} \mathrm{d}\mu(s, t) \quad (\text{all } \mathbf{k} \in \mathbb{Z}^2_+) \text{ [17]}. \tag{2.3}$$

Let $\mathscr{H}$ be a complex Hilbert space and let $\mathscr{B}(\mathscr{H})$ denote the algebra of bounded linear operators on $\mathscr{H}$. For $S, T \in \mathscr{B}(\mathscr{H})$ let $[S, T] := ST - TS$. We say that an n-tuple $\mathbf{T} = (T_1, \ldots, T_n)$ of operators on $\mathscr{H}$ is (jointly) *hyponormal* if the operator matrix $[\mathbf{T}^*, \mathbf{T}] := ([T_j^*, T_i])_{i,j=1}^n$ is positive on the direct sum of n copies of $\mathscr{H}$ (cf. [2,12]). The n-tuple $\mathbf{T}$ is said to be *normal* if $\mathbf{T}$ is commuting and each T_i is normal, and $\mathbf{T}$ is *subnormal* if $\mathbf{T}$ is the restriction of a normal n-tuple to a common invariant subspace. Clearly, normal $\Longrightarrow$ subnormal $\Longrightarrow$ hyponormal. Moreover, the restriction of a hyponormal n-tuple to an invariant subspace is again hyponormal. The Bram–Halmos criterion states that an operator $T \in \mathscr{B}(\mathscr{H})$ is subnormal if and only if the k-tuple $(T, T^2, \ldots, T^k)$ is hyponormal for all $k \geqslant 1$; we say that T is k-hyponormal when the latter condition holds. On the other hand, for a commuting pair $\mathbf{T} \equiv (T_1, T_2)$ of operators on Hilbert space we have

Definition 2.1 (*cf.* [9]). A commuting pair $\mathbf{T} \equiv (T_1, T_2)$ is called *k-hyponormal* if $\mathbf{T}(k) := (T_1, T_2, T_1^2, T_2T_1, T_2^2, \ldots, T_1^k, T_2T_1^{k-1}, \ldots, T_2^k)$ is hyponormal, or equivalently

$$([(T_2^q T_1^p)^*, T_2^n T_1^m])_{\substack{1 \leqslant m+n \leqslant k \\ 1 \leqslant p+q \leqslant k}} \geqslant 0.$$

Clearly, subnormal $\Longrightarrow$ $(k+1)$-hyponormal $\Longrightarrow$ k-hyponormal for every $k \geqslant 1$, and of course 1-hyponormality agrees with the usual definition of joint hyponormality (as above). In [9] we obtained the following multivariable version of the Bram–Halmos criterion for subnormality, which provided an abstract answer to the LPCS, by showing that no matter how k-hyponormal the pair $\mathbf{T}$ might be, it may still fail to be subnormal.

Theorem 2.2 [9, Theorem 2.3]. *Let $\mathbf{T} \equiv (T_1, T_2)$ be a commuting pair of subnormal operators on a Hilbert space $\mathscr{H}$. The following statements are equivalent*:

(i) $\mathbf{T}$ *is subnormal.*
(ii) $\mathbf{T}$ *is k-hyponormal for all $k \in \mathbb{Z}_+$.*

In the single variable case, there are useful criteria for k-hyponormality [6,8]); for 2-variable weighted shifts, a simple criterion for joint hyponormality was given in [5]. The following characterization of k-hyponormality for 2-variable weighted shifts was given in [9, Theorem 2.4].

Theorem 2.3. *Let $\mathbf{T} \equiv (T_1, T_2)$ be a 2-variable weighted shift with weight sequences $\alpha \equiv \{\alpha_{\mathbf{k}}\}$ and $\beta \equiv \{\beta_{\mathbf{k}}\}$. The following statements are equivalent*:

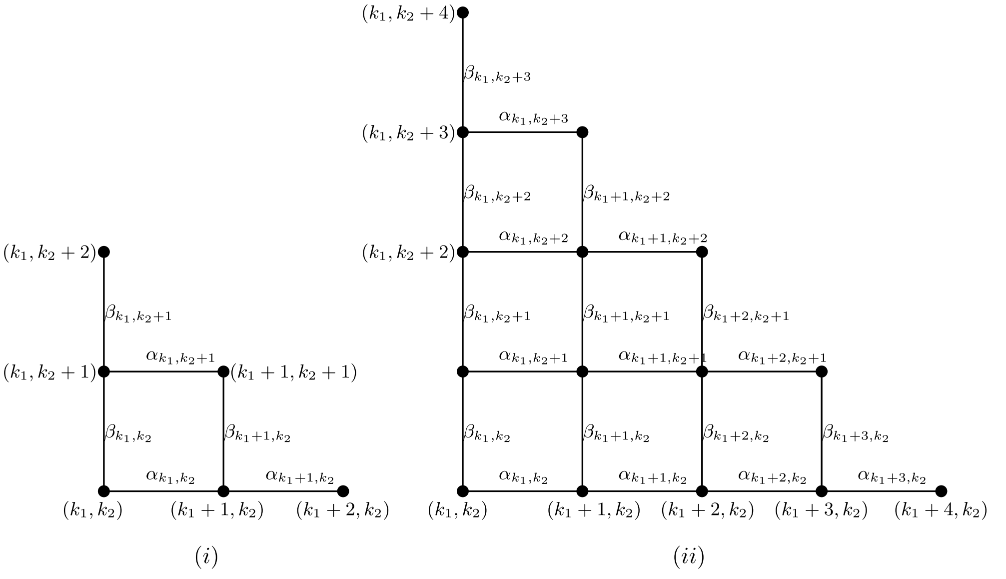

Fig. 1. Weight diagrams used in the Six-point Test and 15-point Test, respectively.

(a) $\mathbf{T}$ *is* k*-hyponormal*.

(b) $M_{\mathbf{u}}(k) := (\gamma_{\mathbf{u}+(m,n)+(p,q)})_{\substack{0\leqslant m+n\leqslant k\\ 0\leqslant p+q\leqslant k}} \geqslant 0$ *for all* $\mathbf{u} \in \mathbb{Z}_+^2$. (*For a subnormal pair* $\mathbf{T}$, *the matrix* $M_{\mathbf{u}}(k)$ *is the truncation of the moment matrix associated to the Berger measure of* $\mathbf{T}$.)

The following special cases of Theorem 2.3 will be essential for our work.

Lemma 2.4 ([5] Six-point Test; cf. Fig. 1(i)). *Let* $\mathbf{T} \equiv (T_1, T_2)$ *be a* 2*-variable weighted shift, with weight sequences* α *and* β. *Then*

$$[\mathbf{T}^*, \mathbf{T}] \geqslant 0 \iff M_{\mathbf{k}}(1) \geqslant 0 \quad (\textit{all } \mathbf{k} \in \mathbb{Z}_+^2)$$
$$\iff \begin{pmatrix} \alpha^2_{\mathbf{k}+\varepsilon_1} - \alpha^2_{\mathbf{k}} & \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} \\ \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} & \beta^2_{\mathbf{k}+\varepsilon_2} - \beta^2_{\mathbf{k}} \end{pmatrix} \geqslant 0 \quad (\textit{all } \mathbf{k} \in \mathbb{Z}_+^2).$$

Lemma 2.5 ([9] 15-point Test; cf. Fig. 1(ii)). *If* $\mathbf{T} \equiv (T_1, T_2)$ *is* 2*-variable weighted shift with weight sequence* $\alpha \equiv \{\alpha_{\mathbf{k}}\}$ *and* $\beta \equiv \{\beta_{\mathbf{k}}\}$, *then* $\mathbf{T}$ *is* 2*-hyponormal if and only if*

$$M_{(k_1,k_2)}(2) \equiv \begin{pmatrix} \gamma_{k_1,k_2} & \gamma_{k_1+1,k_2} & \gamma_{k_1,k_2+1} & \gamma_{k_1+2,k_2} & \gamma_{k_1+1,k_2+1} & \gamma_{k_1,k_2+2} \\ \gamma_{k_1+1,k_2} & \gamma_{k_1+2,k_2} & \gamma_{k_1+1,k_2+1} & \gamma_{k_1+3,k_2} & \gamma_{k_1+2,k_2+1} & \gamma_{k_1+1,k_2+2} \\ \gamma_{k_1,k_2+1} & \gamma_{k_1+1,k_2+1} & \gamma_{k_1,k_2+2} & \gamma_{k_1+2,k_2+1} & \gamma_{k_1+1,k_2+2} & \gamma_{k_1,k_2+3} \\ \gamma_{k_1+2,k_2} & \gamma_{k_1+3,k_2} & \gamma_{k_1+2,k_2+1} & \gamma_{k_1+4,k_2} & \gamma_{k_1+3,k_2+1} & \gamma_{k_1+2,k_2+2} \\ \gamma_{k_1+1,k_2+1} & \gamma_{k_1+2,k_2+1} & \gamma_{k_1+1,k_2+2} & \gamma_{k_1+3,k_2+1} & \gamma_{k_1+2,k_2+2} & \gamma_{k_1+1,k_2+3} \\ \gamma_{k_1,k_2+2} & \gamma_{k_1+1,k_2+2} & \gamma_{k_1,k_2+3} & \gamma_{k_1+2,k_2+2} & \gamma_{k_1+1,k_2+3} & \gamma_{k_1,k_2+4} \end{pmatrix} \geqslant 0$$

for all $\mathbf{k} \in \mathbb{Z}_+^2$.

We now recall the following notation and terminology from [13]:

(i) given a probability measure μ on $X \times Y \equiv \mathbb{R}_+ \times \mathbb{R}_+$, with $\frac{1}{t} \in L^1(\mu)$, the *extremal measure* μ_{ext} (which is also a probability measure) on $X \times Y$ is given by $\mathrm{d}\mu_{\mathrm{ext}}(s,t) := (1 - \delta_0(t))\frac{1}{t\|\frac{1}{t}\|_{L^1(\mu)}}\mathrm{d}\mu(s,t)$; and

(ii) given a measure μ on $X \times Y$, the *marginal measure* μ^X is given by $\mu^X := \mu \circ \pi_X^{-1}$, where $\pi_X : X \times Y \to X$ is the canonical projection onto X. Thus, $\mu^X(E) = \mu(E \times Y)$, for every $E \subseteq X$.

We now list some results which are needed in the proof of Theorem 3.1.

Lemma 2.6 (cf. [21,7, Proposition 2.2]). *Let* $M \equiv \begin{pmatrix} A & B^* \\ B & C \end{pmatrix}$ *be a* 2×2 *operator matrix, where* A *and* C *are square matrices and* B *is a rectangular matrix. Then*

$$M \geqslant 0 \iff \text{there exists } W \text{ such that } \begin{cases} C \geqslant 0, \\ B = CW, \\ A \geqslant W^*CW. \end{cases}$$

For Lemma 2.7, Definition 2.8 and Theorem 3.1 the following two subspaces of $\ell^2(\mathbb{Z}_+^2)$ will be needed: $\mathscr{M} := \bigvee\{e_{\mathbf{k}} : k_2 \geqslant 1\}$ and $\mathscr{N} := \bigvee\{e_{\mathbf{k}} : k_1 \geqslant 1\}$.

Lemma 2.7 ([13] (Subnormal backward extension of a 2-variable weighted shift)). *Consider the* 2*-variable weighted shift whose weight diagram is given in Fig.* 2(i), *and let* $\mathbf{T}|_{\mathscr{M}}$ *denote the restriction of* $\mathbf{T}$ *to* $\mathscr{M}$. *Assume that* $\mathbf{T}|_{\mathscr{M}}$ *is subnormal with associated measure* $\mu_{\mathscr{M}}$, *and that* $W_0 := \mathrm{shift}(\alpha_{00}, \alpha_{10}, \ldots)$ *is subnormal with associated measure* ν. *Then* $\mathbf{T}$ *is subnormal if and only if* (i) $\frac{1}{t} \in L^1(\mu_{\mathscr{M}})$; (ii) $\beta_{00}^2 \leqslant (\|\frac{1}{t}\|_{L^1(\mu_{\mathscr{M}})})^{-1}$; (iii) $\beta_{00}^2\|\frac{1}{t}\|_{L^1(\mu_{\mathscr{M}})}(\mu_{\mathscr{M}})_{\mathrm{ext}}^X \leqslant \nu$. *Moreover*,

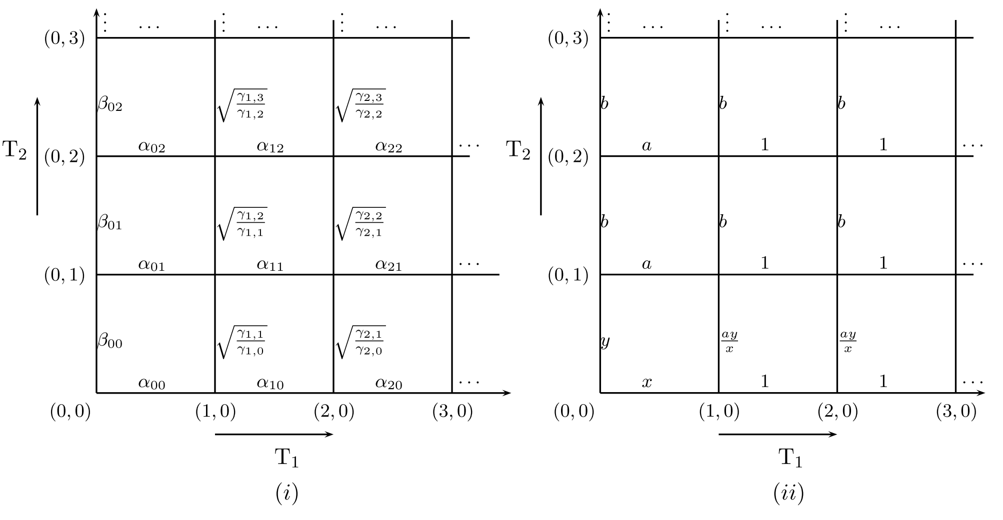

Fig. 2. Weight diagrams of the 2-variable weighted shifts in Lemma 2.7 and Theorem 3.1, respectively.

if $\beta_{00}^2\|\frac{1}{t}\|_{L^1(\mu_{\mathscr{M}})} = 1$, *then* $(\mu_{\mathscr{M}})_{\text{ext}}^X = \nu$. *In the case when* $\mathbf{T}$ *is subnormal*, *the Berger measure* μ *of* $\mathbf{T}$ *is given by*

$$\mathrm{d}\mu(s,t) = \beta_{00}^2\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathscr{M}})} \mathrm{d}(\mu_{\mathscr{M}})_{\text{ext}}(s,t) + \left(\mathrm{d}\nu(s) - \beta_{00}^2\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathscr{M}})} \mathrm{d}(\mu_{\mathscr{M}})_{\text{ext}}^X(s)\right)\mathrm{d}\delta_0(t).$$

Stampfli showed in [22] that for subnormal weighted shifts W_α a *propagation* phenomenon occurs that forces the flatness of W_α whenever two equal weights are present. Later, Curto proved in [6] that a hyponormal weighted shift with *three* equal weights cannot be quadratically hyponormal without being flat: If W_α is quadratically hyponormal and $\alpha_n = \alpha_{n+1} = \alpha_{n+2}$ for some $n \geqslant 0$, then $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$. Choi [3] improved this result, that is, if W_α be quadratically hyponormal and $\alpha_n = \alpha_{n+1}$ for some $n \geqslant 1$, then W_α is *flat*, that is, $W_\alpha \equiv \text{shift}(\alpha_0, \alpha_1, \alpha_1, \ldots)$. In Theorem 2.11, we show that similar propagation phenomena occur for 2-hyponormal 2-variable weighted shifts $\mathbf{T}$. In two variables, the flatness of $\mathbf{T}$ is captured by the so-called *core* of $\mathbf{T}$, $c(\mathbf{T})$ (cf. Definition 2.8 below), as follows: $\mathbf{T}$ is flat when $c(\mathbf{T})$ is a 2-variable weighted shift of tensor form.

Definition 2.8. (i) The *core* of a 2-variable weighted shift $\mathbf{T}$ is the restriction of $\mathbf{T}$ to $\mathscr{M} \cap \mathscr{N}$, in symbols, $c(\mathbf{T}) := \mathbf{T}|_{\mathscr{M}\cap\mathscr{N}}$.

(ii) A 2-variable weighted shift $\mathbf{T}$ is said to be of *tensor form* if $\mathbf{T} \cong (I \otimes W_\alpha, W_\beta \otimes I)$. When $\mathbf{T}$ is subnormal, this is equivalent to requiring that the Berger measure be a Cartesian product $\xi \times \eta$.

(iii) The class of all 2-variable weighted shift $\mathbf{T} \in \mathfrak{H}_0$ whose core is of tensor form will be denoted by $\mathscr{TC}$, that is, $\mathscr{TC} := \{\mathbf{T} \in \mathfrak{H}_0 : c(\mathbf{T}) \text{ is of tensor form}\}$.

We next recall that a 2-variable weighted shift $\mathbf{T}$ is said to be *horizontally flat* when $\alpha_{(k_1,k_2)} = \alpha_{(1,1)}$ for all $k_1, k_2 \geqslant 1$; and we call $\mathbf{T}$ *vertically flat* when $\beta_{(k_1,k_2)} = \beta_{(1,1)}$ for all $k_1, k_2 \geqslant 1$. We say $\mathbf{T}$ is *flat* if $\mathbf{T}$ is horizontally and vertically flat, and that $\mathbf{T}$ is *symmetrically flat* if $\mathbf{T}$ is flat and $\alpha_{(1,1)} = \beta_{(1,1)}$. The next two results show the extent to which propagation holds in the presence of (joint) hyponormality. The first result is implicit in the proof of [15, Theorem 3.3(i)]; we isolate the statement here, and provide a proof for completeness.

Proposition 2.9. *Let* $\mathbf{T}$ *be a commuting hyponormal* 2*-variable weighted shift*, *whose weight diagram is given in Fig.* 2(i). *Assume that* $\alpha_{\mathbf{k}+\varepsilon_1} = \alpha_{\mathbf{k}}$ *for some* $\mathbf{k} \in \mathbb{Z}_+^2$. *Then* $\alpha_{\mathbf{k}+\varepsilon_2} = \alpha_{\mathbf{k}}$ *and* $\beta_{\mathbf{k}+\varepsilon_1} = \beta_{\mathbf{k}}$.

Proof. Recall that, by joint hyponormality, we have

$$\begin{pmatrix} \alpha_{\mathbf{k}+\varepsilon_1}^2 - \alpha_{\mathbf{k}}^2 & \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \beta_{\mathbf{k}}\alpha_{\mathbf{k}} \\ \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \beta_{\mathbf{k}}\alpha_{\mathbf{k}} & \beta_{\mathbf{k}+\varepsilon_2}^2 - \beta_{\mathbf{k}}^2 \end{pmatrix} \geqslant 0.$$

Since $\alpha_{\mathbf{k}+\varepsilon_1} = \alpha_{\mathbf{k}}$, positivity implies that the off-diagonal entry must be zero, i.e.,

$$\alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} = \beta_{\mathbf{k}}\alpha_{\mathbf{k}}. \tag{2.4}$$

By the commuting property (2.1)

$$\alpha_{\mathbf{k}}\beta_{\mathbf{k}+\varepsilon_1} = \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}}. \tag{2.5}$$

Therefore

$$
\begin{aligned}
\alpha^2_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}} &= \alpha_{\mathbf{k}+\varepsilon_2}(\alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}}) \\
&= \alpha_{\mathbf{k}+\varepsilon_2}(\alpha_{\mathbf{k}}\beta_{\mathbf{k}+\varepsilon_1}) \quad \text{(by (2.5))} \\
&= \alpha_{\mathbf{k}}(\alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1}) = \alpha_{\mathbf{k}}(\beta_{\mathbf{k}}\alpha_{\mathbf{k}}) \quad \text{(by (2.4))}.
\end{aligned}
$$

Thus, $\alpha^2_{(\mathbf{k})+\varepsilon_2}\beta_{\mathbf{k}} = \alpha_{\mathbf{k}}(\beta_{\mathbf{k}}\alpha_{\mathbf{k}})$, which implies that $\alpha_{\mathbf{k}+\varepsilon_2} = \alpha_{\mathbf{k}}$. Now (2.5) readily implies that $\beta_{(\mathbf{k})+\varepsilon_1} = \beta_{\mathbf{k}}$, as desired. $\square$

Proposition 2.10 [15, Theorem 4.7(i)]. *Let* $\mathbf{T}$ *be a commuting hyponormal* 2*-variable weighted shift, whose weight diagram is given in Fig.* 2(i). *Assume that* T_1 *is quadratically hyponormal, that* T_2 *is subnormal, and that* $\alpha_{\mathbf{k}+\varepsilon_1} = \alpha_{\mathbf{k}}$ *for some* $\mathbf{k} \in \mathbb{Z}^2_+$. *Then* $\mathbf{T}$ *is horizontally flat.*

In both Propositions 2.9 and 2.10, the basic assumption is that $\mathbf{T}$ is (jointly) hyponormal. On the other hand, under the (stronger) assumption of 2-hyponormality (but without assuming the subnormality of either T_1 or T_2), we can prove that $\mathbf{T}$ is flat whenever $\alpha_{\mathbf{k}+\varepsilon_1} = \alpha_{\mathbf{k}}$ and $\beta_{\mathbf{m}+\varepsilon_2} = \beta_{\mathbf{m}}$ for some $\mathbf{k}$ and $\mathbf{m}$. We do this using the 15-point Test (Lemma 2.5).

Theorem 2.11. *Let* $\mathbf{T}$ *be a commuting* 2*-hyponormal* 2*-variable weighted shift. If* $\alpha_{(k_1,k_2)+\varepsilon_1} = \alpha_{(k_1,k_2)}$ *for some* $k_1, k_2 \geqslant 1$, *then* $\mathbf{T}$ *is horizontally flat. If instead,* $\beta_{(k_1,k_2)+\varepsilon_2} = \beta_{(k_1,k_2)}$ *for some* $k_1, k_2 \geqslant 1$, *then* $\mathbf{T}$ *is vertically flat.*

Proof. Without loss of generality, assume $\alpha_{(k_1,k_2)+\varepsilon_1} = \alpha_{(k_1,k_2)} = 1$. Since $\mathbf{T}$ is 2-hyponormal, we know that T_1 is quadratically hyponormal, and the above mentioned result of Choi implies that $\alpha_{(p,k_2)} = \alpha_{(1,k_2)}$ for all $p \geqslant 1$. An application of Proposition 2.9 now shows that T_1 is of tensor form when restricted to the subspace $\bigvee\{e_{\mathbf{m}} : m_1 \geqslant 1 \text{ and } m_2 \geqslant k_2\}$. If $k_2 = 1$ we are done, so assume $k_2 \geqslant 2$ and consider the matrix

$$
M_{(k_1,k_2-2)}(2) = \begin{pmatrix}
\gamma_{k_1,k_2-2} & \gamma_{k_1+1,k_2-2} & \gamma_{k_1,k_2-1} & \gamma_{k_1+2,k_2-2} & \gamma_{k_1+1,k_2-1} & \gamma_{k_1,k_2} \\
\gamma_{k_1+1,k_2-2} & \gamma_{k_1+2,k_2-2} & \gamma_{k_1+1,k_2-1} & \gamma_{k_1+3,k_2-2} & \gamma_{k_1+2,k_2-1} & \gamma_{k_1+1,k_2} \\
\gamma_{k_1,k_2-1} & \gamma_{k_1+1,k_2-1} & \gamma_{k_1,k_2} & \gamma_{k_1+2,k_2-1} & \gamma_{k_1+1,k_2} & \gamma_{k_1,k_2+1} \\
\gamma_{k_1+2,k_2-2} & \gamma_{k_1+3,k_2-2} & \gamma_{k_1+2,k_2-1} & \gamma_{k_1+4,k_2-2} & \gamma_{k_1+3,k_2-1} & \gamma_{k_1+2,k_2} \\
\gamma_{k_1+1,k_2-1} & \gamma_{k_1+2,k_2-1} & \gamma_{k_1+1,k_2} & \gamma_{k_1+3,k_2-1} & \gamma_{k_1+2,k_2} & \gamma_{k_1+1,k_2+1} \\
\gamma_{k_1,k_2} & \gamma_{k_1+1,k_2} & \gamma_{k_1,k_2+1} & \gamma_{k_1+2,k_2} & \gamma_{k_1+1,k_2+1} & \gamma_{k_1,k_2+2}
\end{pmatrix},
$$

which we know is positive semidefinite since $\mathbf{T}$ is 2-hyponormal (Lemma 2.5). It suffices to prove that $\alpha_{\mathbf{k}-\varepsilon_2} = 1$. Let us focus on the principal submatrix $M \geqslant 0$ determined by rows and columns 1, 3 and 5. Using (2.2) we have

$$
M = \begin{pmatrix}
1 & \beta^2_{\mathbf{k}-2\varepsilon_2} & \beta^2_{\mathbf{k}-2\varepsilon_2}\alpha^2_{\mathbf{k}-\varepsilon_2} \\
\beta^2_{\mathbf{k}-2\varepsilon_2} & \beta^2_{\mathbf{k}-2\varepsilon_2}\beta^2_{\mathbf{k}-\varepsilon_2} & \beta^2_{\mathbf{k}-2\varepsilon_2}\beta^2_{\mathbf{k}-\varepsilon_2} \\
\beta^2_{\mathbf{k}-2\varepsilon_2}\alpha^2_{\mathbf{k}-\varepsilon_2} & \beta^2_{\mathbf{k}-2\varepsilon_2}\beta^2_{\mathbf{k}-\varepsilon_2} & \beta^2_{\mathbf{k}-2\varepsilon_2}\beta^2_{\mathbf{k}-\varepsilon_2}
\end{pmatrix} \geqslant 0. \tag{2.6}
$$

For notational convenience, let $a := \beta^2_{\mathbf{k}-2\varepsilon_2}$, $b := \beta^2_{\mathbf{k}-\varepsilon_2}$ and $c := \alpha^2_{\mathbf{k}-\varepsilon_2}$. Thus

$$
M \equiv \begin{pmatrix} 1 & a & ac \\ a & ab & ab \\ ac & ab & ab \end{pmatrix} \geqslant 0 \iff N := \begin{pmatrix} ab - a^2 & ab - a^2c \\ ab - a^2c & ab - a^2c^2 \end{pmatrix} \geqslant 0.
$$

If $a = b$ then we must necessarily have $c = 1$, that is, $\alpha_{\mathbf{k}-\varepsilon_2} = 1$, as desired. Assume, therefore, that $a < b$. It follows that

$$\begin{aligned} N \geqslant 0 &\iff \det N \geqslant 0 \iff (ab - a^2)(ab - a^2c^2) - (ab - a^2c)^2 \geqslant 0 \\ &\iff -a^3b(c-1)^2 \geqslant 0 \iff c = 1, \end{aligned}$$

that is, $\alpha_{\mathbf{k}-\varepsilon_2} = 1$, as desired. $\square$

3. Proofs of the main results

We are now ready to prove our main results, which we restate for the reader's convenience.

Theorem 3.1. $\mathscr{S} \bigcap \mathfrak{H}_2 = \mathscr{S} \bigcap \mathfrak{H}_\infty$.

Proof. By Propositions 2.9, 2.10, and Theorem 2.11, we can assume, without loss of generality, that $\alpha_{(k_1,k_2)} = \alpha_{(1,0)} = 1$ (all $k_1 \geqslant 1, k_2 \geqslant 0$) and $\beta_{(k_1,k_2)} = \beta_{(0,1)} \leqslant 1$ (all $k_1 \geqslant 0, k_2 \geqslant 1$). For notational convenience, we let $a := \alpha_{(0,1)} < 1$, $b := \beta_{(0,1)} \leqslant 1$, $x := \alpha_{(0,0)} < 1$, and $y := \beta_{(0,0)} < 1$. We summarize this information in Fig. 2(ii).

Let $C_k := \{(a, b, x, y) : \mathbf{T} \in \mathfrak{H}_k\}$ $(k = 1, 2, \infty)$. Clearly, $C_\infty \subseteq C_2 \subseteq C_1$. We first describe concretely the set C_2 in terms of necessary and sufficient conditions on the four parameters a, b, x and y that guarantee the 2-hyponormality of the pair $\mathbf{T}$. We then establish that $C_2 \subseteq C_\infty$. Finally, in Theorem 3.3 we will show that $C_2 \subsetneq C_1$.

It is straightforward to verify that $\mathbf{T}|_{\mathscr{M}} \cong (I \otimes \mathrm{shift}(a, 1, 1, \ldots), bU_+ \otimes I)$ and $\mathbf{T}|_{\mathscr{N}} \cong (I \otimes U_+, \mathrm{shift}(\frac{ay}{x}, b, b, \ldots) \otimes I)$; thus, $\mathbf{T}|_{\mathscr{M}}$ and $\mathbf{T}|_{\mathscr{N}}$ are subnormal. As a consequence, to describe C_2 we only need to apply the 15-point Test (Lemma 2.5) at $\mathbf{k} = (0, 0)$, that is, we need to guarantee that $M_{(0,0)}(2) \geqslant 0$. We thus have

$$\mathbf{T} \in \mathfrak{H}_2 \iff M_{(0,0)}(2) \geqslant 0.$$

Now, since the moments $\gamma_{\mathbf{k}}$ ($\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2$) associated with $\mathbf{T}$ are

$$\gamma_{\mathbf{k}} = \begin{cases} 1 & \text{if } k_1 = 0 \text{ and } k_2 = 0, \\ x^2 & \text{if } k_1 \geqslant 1 \text{ and } k_2 = 0, \\ y^2 b^{2(k_2-1)} & \text{if } k_1 = 0 \text{ and } k_2 \geqslant 1, \\ a^2 y^2 b^{2(k_2-1)} & \text{if } k_1 \geqslant 1 \text{ and } k_2 \geqslant 1, \end{cases}$$

it follows that

$$\begin{aligned} M_{(0,0)}(2) \equiv &\begin{pmatrix} 1 & x^2 & y^2 & x^2 & a^2y^2 & b^2y^2 \\ x^2 & x^2 & a^2y^2 & x^2 & a^2y^2 & a^2b^2y^2 \\ y^2 & a^2y^2 & b^2y^2 & a^2y^2 & a^2b^2y^2 & b^4y^2 \\ x^2 & x^2 & a^2y^2 & x^2 & a^2y^2 & a^2b^2y^2 \\ a^2y^2 & a^2y^2 & a^2b^2y^2 & a^2y^2 & a^2b^2y^2 & a^2b^4y^2 \\ b^2y^2 & a^2b^2y^2 & b^4y^2 & a^2b^2y^2 & a^2b^4y^2 & b^6y^2 \end{pmatrix} \geqslant 0 \\ \iff M := &\begin{pmatrix} 1 & x^2 & y^2 & a^2y^2 \\ x^2 & x^2 & a^2y^2 & a^2y^2 \\ y^2 & a^2y^2 & b^2y^2 & a^2b^2y^2 \\ a^2y^2 & a^2y^2 & a^2b^2y^2 & a^2b^2y^2 \end{pmatrix} \geqslant 0 \end{aligned}$$

(since the sixth row is a multiple of the third row, and the second and fourth rows are identical). If we now interchange the second and third rows and columns, we see that the positivity of M is determined by the positivity of

$$\begin{pmatrix} \begin{pmatrix} 1 & y^2 \\ y^2 & b^2y^2 \end{pmatrix} & \begin{pmatrix} x^2 & a^2y^2 \\ a^2y^2 & a^2b^2y^2 \end{pmatrix} \\ \begin{pmatrix} x^2 & a^2y^2 \\ a^2y^2 & a^2b^2y^2 \end{pmatrix} & \begin{pmatrix} x^2 & a^2y^2 \\ a^2y^2 & a^2b^2y^2 \end{pmatrix} \end{pmatrix} =: \begin{pmatrix} A & B \\ B & B \end{pmatrix}.$$

Thus, from Lemma 2.6 (with $C = B$ and $W = I$), we have

$$M \geqslant 0 \Longleftrightarrow A \geqslant B \geqslant 0.$$

Now observe that $B \geqslant 0 \iff ay \leqslant bx$ and

$$A - B = \begin{pmatrix} 1 - x^2 & y^2(1-a^2) \\ y^2(1-a^2) & b^2y^2(1-a^2) \end{pmatrix}.$$

Since the (1,1)-entry of $A - B$ is always positive, the positivity of $A - B$ is completely determined by its determinant; that is

$$\begin{aligned} A - B \geqslant 0 &\iff \det(A-B) \geqslant 0 \\ &\iff y^2(1-a^2) \leqslant b^2(1-x^2). \end{aligned}$$

It follows that

$$\mathbf{T} \in \mathfrak{H}_2 \iff ay \leqslant bx \quad \text{and} \quad y^2(1-a^2) \leqslant b^2(1-x^2). \tag{3.1}$$

We thus see that

$$C_2 = \{(a,b,x,y) : 0 < x < 1, 0 < y < 1, ay \leqslant bx, \text{ and } y^2(1-a^2) \leqslant b^2(1-x^2)\}. \tag{3.2}$$

We will now prove that $C_2 \subseteq C_\infty$. Let $(a,b,x,y) \in C_2$. Let $z := \frac{ay}{x}$.
Case 1. If $z = b$, then $\mathbf{T}|_{\mathcal{N}} \equiv (I \otimes U_+, bU_+ \otimes I)$, and a straightforward application of Lemma 2.7 in the s direction shows that $\mathbf{T}$ is subnormal if and only if $bx \leqslant y$ if and only if $a \leqslant 1$ (since $bx = ay$), which is true. Thus, $(a,b,x,y) \in C_\infty$.

Case 2. Assume now that $z < 1$. Since $\mathbf{T}|_{\mathcal{N}} \equiv (I \otimes U_+, bS_{z/b} \otimes I)$ is subnormal with Berger measure $\mu_{\mathcal{N}} \equiv \delta_1 \times [(1 - \frac{z^2}{b^2})\delta_0 + \frac{z^2}{b^2}\delta_{b^2}]$, we can think of $\mathbf{T}$ as a backward extension of $\mathbf{T}|_{\mathcal{N}}$ (in the s direction) and apply Lemma 2.7. Note that $\|\frac{1}{s}\|_{L^1(\mu_{\mathcal{N}})} = 1$, so $\mathrm{d}(\mu_{\mathcal{N}})_{\text{ext}}(s,t) \equiv (1 - \delta_0(s))\frac{1}{s}\mathrm{d}\mu_{\mathcal{N}}(s,t) = \mathrm{d}\delta_1(s)[(1 - \frac{z^2}{b^2})\mathrm{d}\delta_0(t) + \frac{z^2}{b^2}d\delta_{b^2}(t)]$ and $\alpha_{00}^2\|\frac{1}{s}\|_{L^1(\mu_{\mathcal{N}})}(\mu_{\mathcal{N}})_{\text{ext}}^Y = x^2[(1 - \frac{z^2}{b^2})\delta_0 + \frac{z^2}{b^2}\delta_{b^2}]$. Thus, by Lemma 2.7

$$\mathbf{T} \text{ is subnormal} \iff \alpha_{00}^2 \left\| \frac{1}{s} \right\|_{L^1(\mu_{\mathcal{N}})} (\mu_{\mathcal{N}})_{\text{ext}}^Y \leqslant \eta_0$$

(where η_0 denotes the Berger measure of $\operatorname{shift}(y,b,b,\ldots)$)

$$\iff x^2\left[\left(1 - \frac{z^2}{b^2}\right)\delta_0 + \frac{z^2}{b^2}\delta_{b^2}\right] \leqslant \left(1 - \frac{y^2}{b^2}\right)\delta_0 + \frac{y^2}{b^2}\delta_{b^2}$$

$$\iff x^2\left(1-\frac{z^2}{b^2}\right) \leqslant \left(1-\frac{y^2}{b^2}\right) \text{ and } \frac{x^2z^2}{b^2} \leqslant \frac{y^2}{b^2}$$
$$\iff y^2(1-a^2) \leqslant b^2(1-x^2) \tag{3.3}$$

as in the last condition in (3.2). Therefore, **T** is subnormal, and the proof is complete. □

Theorem 3.2. *Let* $\mathbf{T} \equiv (T_1, T_2) \in \mathcal{S}$, *with weight diagram given by Fig.* 2(ii), *and assume that* **T** *is* 2*-hyponormal. Then* **T** *is subnormal, with Berger measure given as*

$$\mu = \frac{1}{b^2}\Big\{[b^2(1-x^2) - y^2(1-a^2)]\delta_{(0,0)} + y^2(1-a^2)\delta_{(0,b^2)} + (b^2x^2 - a^2y^2)\delta_{(1,0)} + a^2y^2\delta_{(1,b^2)}\Big\}. \tag{3.4}$$

Proof. We apply the main result in [11], in the special form needed here; cf. [11, Proposition 3.1]. For a 2-variable subnormal weighted shift with weight diagram given by Fig. 2(ii), the Berger measure is

$$\mu = \varphi \times (\delta_0 - \delta_{b^2}) + y^2 \left\|\frac{1}{t}\right\|_{L^1(\psi)} \delta_0 \times (\tilde{\psi} - \delta_{b^2}) + \xi_x \times \delta_{b^2}, \tag{3.5}$$

where $\psi = (1-a^2)\delta_{b^2}, \varphi = \xi_x - y^2\frac{1-a^2}{b^2}\delta_0 - \frac{a^2y^2}{b^2}\delta_1, \mathrm{d}\tilde{\psi}(t) := \frac{1}{t\|\frac{1}{t}\|_{L^1(\psi)}}d\psi(t)$ and ξ_x is the Berger measure of S_x. A straightforward calculation shows that $\tilde{\psi} = \delta_{b^2}$, so that (3.5) becomes

$$\mu = \left\{\left[\frac{b^2(1-x^2) - y^2(1-a^2)}{b^2}\right]\delta_0 + \frac{b^2x^2 - a^2y^2}{b^2}\delta_1\right\} \times (\delta_0 - \delta_{b^2}) + [(1-x^2)\delta_0 + x^2\delta_1] \times \delta_{b^2},$$

which easily leads to the desired formula (3.4). □

We conclude this section with a proof of Theorem 1.4, which we reformulate in terms of C_1 and C_2. Toward this end, we need a description of $C_1 := \{(a, b, x, y) : \mathbf{T} \in \mathfrak{H}_1\}$: by the Six-point Test (Lemma 2.4), **T** is hyponormal if and only if

$$M_{(0,0)}(1) \equiv \begin{pmatrix} 1 & x^2 & y^2 \\ x^2 & x^2 & a^2y^2 \\ y^2 & a^2y^2 & b^2y^2 \end{pmatrix} \geqslant 0.$$

Theorem 3.3. $C_2 \subsetneq C_1$.

Proof. Since $x < 1$, $M_{(0,0)}(1) \geqslant 0$ if and only if $\det M_{(0,0)}(1) \geqslant 0$, that is, if and only if

$$P_1 := y^2(b^2x^2 - a^4y^2 - b^2x^4 + 2a^2x^2y^2 - x^2y^2) \geqslant 0.$$

Let $x > \frac{\sqrt{2}}{2}$ and let $a := \sqrt{2x^2-1}$. It follows that $1 - a^2 = 2(1-x^2)$, so that

$$P_2 := b^2(1-x^2) - y^2(1-a^2) = b^2(1-x^2) - 2y^2(1-x^2) = (b^2 - 2y^2)(1-x^2)$$

and it suffices to choose y such that $\frac{\sqrt{2}}{2}b < y < b$ to make $P_2 < 0$, and thus break 2-hyponormality (cf. (3.1)). On the other hand

$$\begin{aligned} P_1 \equiv & P_2x^2y^2 + y^4a^2(x^2 - a^2) \\ & = y^2(b^2x^2 - b^2x^4 - y^2 + x^2y^2) \\ & = (b^2x^2 - y^2)(1 - x^2), \end{aligned}$$

which can be made nonnegative by taking x close to 1. This shows that $C_2 \subsetneq C_1$. $\square$

Acknowledgments

The authors are grateful to the referee for helpful suggestions that improved the presentation. Some of the results in this paper were motivated by calculations done with the software tool *Mathematica* [23].

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